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The viscous damping of cnoidal waves

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The viscous damping of cnoidal waves progressing over a smooth horizontal bed is investigated. First approximations are derived for the attenuation of wave height with distance and for the friction coefficient at the bed. Attenuation coefficients are larger than those predicted on the basis of shallow-water sinusoidal wave theory and, unlike the case of sinusoidal waves, they are not independent of wave height. The limiting case of the solitary wave, considered previously by Keulegan (1948), is also discussed.

1. Introduction

The viscous damping of progressive surface waves is a topic of widespread interest and practical importance. Biesel (1949) investigated the case of two-dimensional periodic sinusoidal waves in water of constant and finite depth and calculated the attenuation coefficient corresponding to an exponential decay of wave height. Hunt (1952) subsequently extended this theory to include the case of waves propagating in a channel of finite width over a gently sloping bed and derived a modified expression for the attenuation coefficient. Hunt (1964) later considered a higher approximation to the exponential decay of two-dimensional sinusoidal waves, such that the theory extended to the deep-water case in which energy dissipation occurs principally in the fluid interior.

On the other hand, the damping of solitary waves was investigated earlier by Keulegan (1948), who found that the wave-height attenuation is not exponential but rather follows an inverse power law. The theories of Biesel (1949) and Hunt (1952, 1964), being based on sinusoidal wave theory, are valid in shallow water only for waves of very small height and in any case do not predict the solitary-wave limit.

Since the shallow-water range is of particular interest, a theory of cnoidal-wave damping is clearly relevant and is the objective of the present paper. The method employed here, used by Hunt (1952) for sinusoidal waves, is based on equating the energy dissipation within the fluid to the spatial decrease in energy flux in the direction of wave propagation. Recently Isaacson (1976) has derived the velocity distribution within the bottom laminar boundary layer of cnoidal waves in the course of investigating the mass-transport velocity. A knowledge of this velocity distribution permits the calculation of energy dissipation within the boundary layer, thus enabling a theory of viscous damping to be developed for cnoidal waves. As expected, the two extreme cases of cnoidal-wave damping correspond to sinusoidal-wave damping in shallow water (Biesel 1949) and to solitary-wave damping (Keulegan 1948). The bottom friction is of associated interest and is also calculated for cnoidal waves.

2. Theoretical development

The water is assumed incompressible and to have an uncontaminated free surface. The wave motion is assumed to be time periodic and two-dimensional and to propagate over a smooth horizontal bed. The attenuation, then, is considered to occur with distance in the direction of wave propagation as is generally the case in laboratory experiments.

Although the wave characteristics vary in the direction of propagation, the wave period T and still water depth d are taken as invariant for a particular wave train. Results of cnoidal wave theory (Laitone 1960) are expressed as power series in the parameter H/h, where H is the wave height and h the trough depth. However, since h itself is expected to vary with distance, we shall here prefer to use the alternative parameter $\epsilon = H/d$ and the behaviour of ϵ will then directly reflect that of H.

Let x denote the co-ordinate in the direction of wave propagation with origin fixed relative to the bed, y the vertical co-ordinate measured upwards from the bed, U and u the horizontal velocity components in the fluid interior and within the bottom boundary layer respectively, and t time. Also let ρ denote the fluid density, ν the kinematic fluid viscosity and μ $(=\rho\nu)$ the dynamic fluid viscosity.

The horizontal fluid velocity U based on Laitone's (1960) cnoidal theory is given in the co-ordinate system adopted here in terms of the Jacobian elliptic function cn with argument $q = K(\kappa)\,\pi^{-1}(kx-\omega t)$ and modulus κ as

$$\frac{U}{(gh)^{\frac{1}{2}}} = \left(\frac{H}{h}\right)\left\{\mathrm{cn}^2(q) - \left(\frac{\gamma-\kappa'^2}{\kappa^2}\right)\right\} + O\left[\left(\frac{H}{h}\right)^2\right]. \tag{2.1}$$

Here γ is the ratio $E(\kappa)/K(\kappa)$, $E(\kappa)$ is the complete elliptic integral of the second kind, $K(\kappa)$ is the complete elliptic integral of the first kind and $\kappa'^2 = 1-\kappa^2$. Also k is the wavenumber $(=2\pi/L)$, where L is the wavelength, ω the wave angular frequency $(=2\pi/T)$ and g the acceleration due to gravity. It is convenient to follow the approach of Isaacson (1976) and represent this velocity as a complex Fourier series:

$$\left.\begin{aligned}\frac{U}{(gh)^{\frac{1}{2}}} &= \left(\frac{H}{h}\right)\sum_{n=-\infty}^{\infty} A'_n e^{in\theta} + O\left[\left(\frac{H}{h}\right)^2\right],\\ \text{with}\qquad A'_{-n} &= A'_n, \quad A'_0 = 0.\end{aligned}\right\} \tag{2.2}$$

Here $\theta = kx-\omega t$. It was indicated how A'_n may be determined numerically for all n for any value of the modulus κ.

Isaacson (1976) calculated the velocity within the laminar boundary layer at the bed to a first approximation:

$$\frac{u}{(gh)^{\frac{1}{2}}} = \left(\frac{H}{h}\right)\sum_{n=-\infty}^{\infty} A'_n[1-\exp(-\alpha_n y/\delta)]\,e^{in\theta} + O\left[\left(\frac{H}{h}\right)^2\right], \tag{2.3}$$

where $\delta = (2\nu/\omega)^{\frac{1}{2}}$, the boundary-layer thickness, and

$$\alpha_n = (1-i)\, n^{\frac{1}{2}}. \tag{2.4}$$

The rate of energy dissipation in an incompressible fluid may be expressed in terms of the Rayleigh dissipation function and, assuming the boundary layer to be thin (relative to both h and L), the predominant contribution to the dissipation rate derives from the velocity gradient $\partial u/\partial y$ within the boundary layer. As a boundary-layer approximation, then, the average rate of energy dissipation per unit length in the x direction (and per unit width) is

$$\overline{\frac{\partial E_L}{\partial t}} = \frac{\mu}{L}\int_0^L \int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy\, dx. \tag{2.5}$$

The above expression may be evaluated by substituting for u from (2.3). Since d is given by cnoidal theory as $h(1+O[\epsilon])$, we may replace h by d to the present order of approximation in order to work with the parameter $\epsilon = H/d$ as mentioned earlier. Furthermore, A_n' depends only on the modulus κ and we eventually have

$$\overline{\partial E_L/\partial t} = \mu g d \delta^{-1} \epsilon^2 f_1(\kappa) + O[\epsilon^3], \tag{2.6}$$

where
$$f_1(\kappa) = 2\sum_{n=1}^{\infty} n^{\frac{1}{2}} A_n'^2. \tag{2.7}$$

The rate of energy transfer per unit width across a plane of constant x may be written in terms of the wave speed c $(=\omega/k)$ as

$$\frac{\partial E}{\partial t} = \rho c \int_0^{h+\eta} U^2 dy, \tag{2.8}$$

where η is the free-surface elevation above the trough level $y = h$. The above equation refers to the irrotational motion of the fluid interior and derives from the alternative expression generally used for finite amplitude waves (see, for example, Longuet-Higgins 1975). Substituting for U from (2.1), taking the time average, and replacing h by d for the same reasons as mentioned earlier, we obtain for the average rate of energy transfer

$$\overline{\partial E/\partial t} = \rho c g d^2 \epsilon^2 f_2(\kappa) + O[\epsilon^3], \tag{2.9}$$

where
$$f_2(\kappa) = 2\sum_{n=1}^{\infty} A_n'^2 = \frac{1}{3\kappa^4}\{2\gamma(2-\kappa^2) - 3\gamma^2 - \kappa'^2\}. \tag{2.10}$$

By the principle of energy conservation, the difference between the average rates at which energy crosses two planes a short distance apart is due to the dissipation between those planes. That is

$$\frac{\partial}{\partial x}\left(\overline{\frac{\partial E}{\partial t}}\right) = -\overline{\frac{\partial E_L}{\partial t}}. \tag{2.11}$$

Now c is given by cnoidal theory as $(gh)^{\frac{1}{2}}(1+O[\epsilon])$ and also d and T are invariant with x. Thus substituting (2.6) and (2.9) into (2.11), omitting terms of order ϵ^3 and rearranging gives

$$f_2(\kappa)\frac{\partial \epsilon}{\partial x} + \tfrac{1}{2}\epsilon\frac{\partial f_2(\kappa)}{\partial x} = -\frac{\epsilon\beta}{d} f_1(\kappa), \tag{2.12}$$

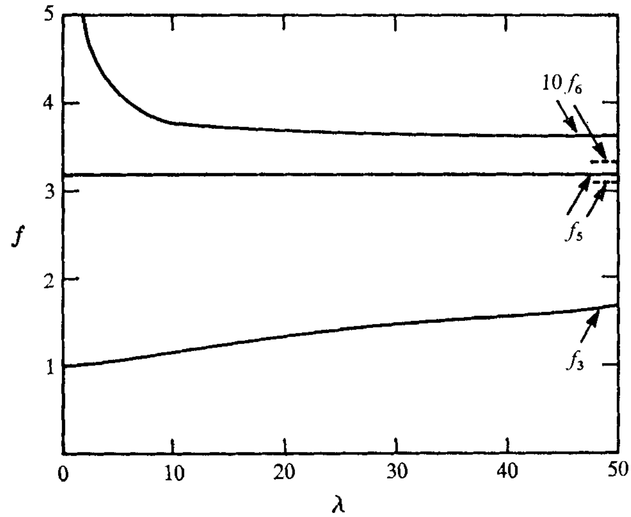

FIGURE 1. Dependence of certain functions on the parameter λ. ---, limiting values as $\lambda \to \infty$.

in which $$\beta = \nu/2g^{\frac{1}{2}}d^{\frac{1}{2}}\delta. \tag{2.13}$$

A consequence of taking T and d constant is that a unique relationship exists between ϵ and κ. We may then consider the modulus κ as a function of ϵ for a particular wave train and thus in turn as a function of x. Therefore

$$\frac{\partial f(\kappa)}{\partial x} = \frac{\partial f(\epsilon)}{\partial x} = \frac{\partial f(\epsilon)}{\partial \epsilon}\frac{\partial \epsilon}{\partial x}. \tag{2.14}$$

Substituting in (2.12) gives

$$\frac{1}{\epsilon f_3(\epsilon)}\frac{\partial \epsilon}{\partial x} = -\frac{\beta}{d}, \tag{2.15}$$

where $$f_3(\epsilon) = \left[\frac{f_2}{f_1} + \frac{\epsilon}{2f_1}\frac{\partial f_2}{\partial \epsilon}\right]^{-1}. \tag{2.16}$$

We shall find it convenient to use the parameter $\lambda = \kappa^2 K^2(\kappa)$ rather than κ itself since to the first approximation cnoidal theory gives

$$\frac{d}{gT^2} = \frac{3\epsilon}{16\kappa^2 K^2(\kappa)} = \frac{3\epsilon}{16\lambda}, \tag{2.17}$$

and thus λ is directly proportional to ϵ for a given train. The variation of f_3 with λ may be readily calculated and is given in figure 1.

The integration of (2.15) may be carried out numerically for any given value of d/gT^2 to obtain $\epsilon(x)$, but some initial rearrangement is desirable since the integrand becomes very large for small ϵ. Furthermore, from sinusoidal wave theory we expect the decay to be closely exponential and we therefore rearrange (2.15) and integrate to give

$$\ln\left(\frac{\epsilon}{\epsilon_0}\right) + \int_{\epsilon_0}^{\epsilon} \frac{\lambda}{\epsilon} f_4 \, d\epsilon = -\frac{\beta x}{d}, \tag{2.18}$$

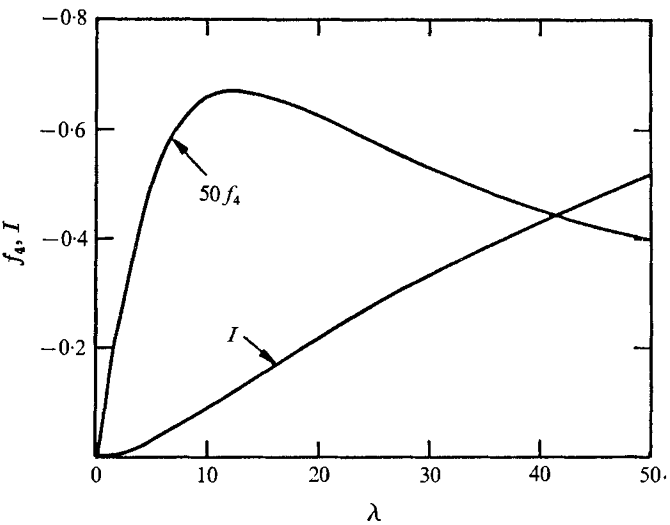

FIGURE 2. Variation of $f_4(\lambda)$ and the integral $I(\lambda)$ used to calculate wave-height attenuation.

where ϵ_0 is the value of ϵ at the origin $x = 0$ and

$$f_4 = \frac{1}{\lambda}\left(\frac{1}{f_3} - 1\right) = \frac{1}{\lambda}\left(\frac{f_2}{f_1} + \frac{\epsilon}{2f_2}\frac{\partial f_2}{\partial \epsilon} - 1\right). \tag{2.19}$$

Now since ϵ is proportional to λ for any given train we have

$$\int_{\epsilon_0}^{\epsilon} \frac{\lambda}{\epsilon} f_4 d\epsilon = \int_{\lambda_0}^{\lambda} f_4 d\lambda = I(\lambda) - I(\lambda_0), \tag{2.20}$$

in which λ_0 is the value of λ corresponding to ϵ_0. The variation of I with λ has been calculated numerically and $I(\lambda)$, together with $f_4(\lambda)$, is presented in figure 2. Writing λ as $\epsilon\lambda_0/\epsilon_0$, we finally have in place of (2.18)

$$\ln\left(\frac{\epsilon}{\epsilon_0}\right) + I\left(\frac{\epsilon}{\epsilon_0}\lambda_0\right) - I(\lambda_0) = -\frac{\beta x}{d}. \tag{2.21}$$

This, then, expresses the profile $\epsilon(x)$ in a general form as the variation of ϵ/ϵ_0 with $\beta x/d$ for any given value of λ_0 ($= 3\epsilon_0 g T^2/16d$). Profiles for the limiting case $\lambda_0 \to 0$ and for $\lambda_0 = 20$ and $\lambda_0 = 50$ are presented in figure 3. The limiting case $\lambda_0 \to 0$ (corresponding to $\kappa, \epsilon \to 0$) reduces to the exponential damping encountered with sinusoidal waves.

In order to compare more precisely the present results with the exponential decay involved in previous work, we define a dimensionless attenuation coefficient a as

$$a = -\frac{d}{\epsilon}\frac{\partial \epsilon}{\partial x} = -\frac{d}{H}\frac{\partial H}{\partial x}. \tag{2.22}$$

In the present case this is not independent of wave height and will therefore vary slowly with x. From (2.15), we have

$$a = \beta f_3(\kappa), \tag{2.23}$$

where $f_3(\lambda)$ is given in figure 1.

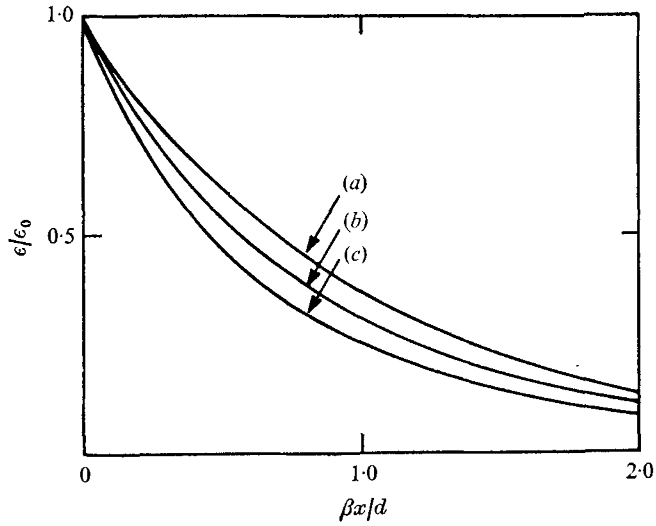

FIGURE 3. Attenuation of wave height with distance in the direction of wave propagation indicated by ϵ/ϵ_0 as a function of $\beta x/d$. (a) $\lambda_0 = 0$, (b) $\lambda_0 = 20$, (c) $\lambda_0 = 50$.

The bottom friction is a feature of associated interest which has been measured experimentally by several authors and the calculation for cnoidal waves follows directly from (2.3). Following Eagleson (1962), we define a characteristic friction coefficient C_f and wave Reynolds number Re as

$$C_f = \overline{|\tau_0|}/\tfrac{1}{2}\rho\overline{U^2}, \quad Re = \pi\overline{U^2}/\omega\nu, \tag{2.24}$$

where an overbar denotes a temporal mean, U is here the velocity at the outer edge of the boundary layer and τ_0 $(=\mu(\partial u/\partial y)_{y=0})$ is the shear stress at the bed. We eventually find the relation between these to be

$$C_f = f_5(\kappa)\,Re^{-\frac{1}{2}}, \tag{2.25}$$

in which

$$f_5(\kappa) = \left(\frac{2\pi}{f_2(\kappa)}\right)^{\frac{1}{2}}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=-\infty}^{\infty}\alpha_n A_n' e^{in\theta}\right|d\theta\right\}. \tag{2.26}$$

The variation of f_5 with λ has been computed and is included in figure 1.

3. The limiting values of κ

By expressing the functions of κ previously encountered as power series in κ^2, the limiting values of these as κ tends to zero may be determined. Thus, as κ (and ϵ) approaches zero, we have for the attenuation coefficient from (2.23)

$$a = \beta, \tag{3.1}$$

a result independent of wave height. The result given by Biesel (1949) for sinusoidal waves reduces in shallow water to $\frac{1}{4}k\delta$. From the definition of β, equation (2.13), we see that (3.1) duplicates the result corresponding to shallow-water sinusoidal wave theory, as expected. As κ departs from the lower limit of zero, the attenuation coefficient a increases relative to the shallow-water sinusoidal-wave prediction according to $f_3(\kappa)$. Regarding the bottom friction, the limiting value of f_5 as $\kappa \to 0$ is $4(2/\pi)^{\frac{1}{2}}$, which was previously obtained by Eagleson (1962, equation (38)) for sinusoidal waves.

In very shallow water as κ approaches unity, k becomes very small, T and, therefore, δ become very large, and it is more appropriate to express the attenuation coefficient a in terms of variables which remain finite as the solitary-wave case is approached.

Rearrangement of (2.23) gives

$$a = \frac{\nu^{\frac{1}{2}} H^{\frac{1}{4}}}{g^{\frac{1}{4}} d} f_6(\kappa), \tag{3.2}$$

where

$$f_6(\kappa) = \frac{\pi^{\frac{1}{2}} 3^{\frac{1}{4}}}{4} \frac{f_3(\kappa)}{(\kappa K(\kappa))^{\frac{1}{2}}}. \tag{3.3}$$

The variation of f_6 with λ is given in figure 1.

When $K(\kappa)$ is very large, κ is approximately $1/K(\kappa)$, and we have from (2.10)

$$K(\kappa) f_2(\kappa) \simeq \tfrac{2}{3}. \tag{3.4}$$

The limiting value of $f_1(\kappa)$ may be obtained by referring to Keulegan's (1948) derivation of the rate of total energy dissipation for the solitary wave [his equation (42)]. The rate of energy dissipation over a wavelength for cnoidal waves is given by (2.6) multiplied by L, and comparing the limiting case of cnoidal waves as $K(\kappa)$ becomes very large with Keulegan's (1948) expression, we obtain

$$K^{\frac{1}{2}}(\kappa) f_1(\kappa) \simeq 2^{\frac{3}{2}} N/\pi, \tag{3.5}$$

where N is an integral which Keulegan (1948) estimated to have the value 0·316. From (3.4) and (3.5) we have finally for $\kappa \to 1$

$$f_6 = 3^{\frac{1}{4}} 2^{\frac{1}{2}} N/\pi^{\frac{1}{2}} \simeq \tfrac{1}{3}, \tag{3.6}$$

which corresponds to the expression obtained by Keulegan (1948). This is expected since his expression for energy dissipation has been used here and the arguments used in § 2 may be rephrased in terms of total energy dissipation over a wave period (as Hunt (1952) has done for sinusoidal waves), in which case the solitary-wave limit will hold good. We note that the inverse power law for wave-height attenuation found by Keulegan (1948) now follows immediately from (2.22) and (3.2).

The relation between the friction coefficient and Reynolds number for κ approaching unity may be considered in a similar manner. Although the definitions of C_f and Re adopted here are somewhat artificial for application to solitary waves, they may be written in terms of integrals over a wave period in order to avoid the use of temporal mean values which approach zero. The limit of f_5 corresponding to the solitary wave has been evaluated numerically by Keulegan's (1948) approach and found to take the value 3·09. The limits of f_5 and f_6 as $\kappa \to 1$ $(\lambda \to \infty)$ are indicated in figure 1.

It is pointed out that, although Hunt (1952) compared the shallow-water limit of his expression for a with Keulegan's (1948) results for solitary-wave damping, the comparison was in fact made with the damping of a solitary rectangular travelling wave of finite length. The attenuation coefficient [defined by

(2.22)] predicted on the basis of shallow-water sinusoidal wave theory is $\frac{1}{2}k\delta$, which, for any given d, tends to zero as the solitary-wave limit is approached ($k, kd \to 0$). This result is contrary to Keulegan's (1948) result $f_6 = \frac{1}{3}$, which does correspond to the limit of cnoidal waves.

4. Discussion

The present theory is based on only the first approximation to cnoidal waves, and may thus be expected to remain valid only for shallow-water waves, say with a wave depth parameter kd up to about 0·3. Previous experimental investigations involving the measurement of attenuation coefficients (e.g. Eagleson 1962; Lukasik & Grosch 1963; Treloar & Brebner 1970) have for the most part covered a higher range of kd. Attenuation coefficients have usually been estimated by initially assuming an exponential decay of height, thus disregarding the variation of a with height itself, and furthermore the effects of surface tension, not considered in the present study, are generally appreciable. It follows that a comparison of the present theory with available experimental data is inappropriate. Nevertheless, in the case of shallower waves, the present approach suggests a further contributory effect accounting for the large attenuation coefficients previously measured. For example, with $kd = 0{\cdot}25$ and $\epsilon = 0{\cdot}2$ the present theory predicts a to be almost 40 % larger than that given by sinusoidal wave theory.

The result showing that C_f varies as $Re^{-\frac{1}{2}}$ for cnoidal waves is hardly surprising since this power law is merely a consequence of the boundary-layer approximation and is independent of the particular flow beyond the boundary layer. Isaacson & Isaacson (1975, p. 108) have indicated that the laminar boundary-layer equations are dimensionally homogeneous in the extended set of reference dimensions $MXYZT$, where X, Y and Z are the length scales in the three orthogonal directions x, y and z respectively, and consequently a dimensional analysis based on this set of dimensions directly yields the above power law. In the case of cnoidal waves, where the characteristic velocity variation outside the boundary layer depends on the modulus κ, the proportionality constant is strictly a function of κ and we therefore duplicate (2.25), in which of course the precise form of $f_5(\kappa)$ is not to be obtained by dimensional methods.

Finally, we may note that the wave depth parameter kd is given by cnoidal theory as

$$kd = \frac{\pi(3\epsilon)^{\frac{1}{2}}}{2\kappa K(\kappa)}(1 + O[\epsilon]). \tag{4.1}$$

Comparing this equation with (2.17), we see that to the first approximation kd and therefore L remain constant for a given train. This is in spite of the first-order dependence of L on wave height, and the general assertion that L itself does vary in very shallow water. To the second approximation, of course, L is expected to vary with x.

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